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ABSTRACT

The author claims the development of an improved method for teaching basic addition in the elementary schools. Two advantages of the method are (1) more effective grouping of basic addition facts, and systematic and consistent use of reasoning in their derivation, and (2) use of a special classroom technique to improve the proficiency of a child in the application of basic arithmetic facts. An analysis is presented to show how the organization of various methods for teaching addition came into use, and compares their advantages and disadvantages. (RP)

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AN IMPROVED METHOD FOR TEACHING BASIC ADDITION IN ELEMENTARY SCHOOLS

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(A short biographical sketch of the author appears on page 15 of this paper)

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ABSTRACT

The paper reports the development of an improved method for teaching basic addition in elementary schools. This improved method can be used in the arithmetic books for first and second grades. The adoption of the method will not require any change in the arithmetic books for upper grades, unless it is desired to upgrade the curriculum and the standard of instruction, nor, will it require new or special training for teachers. Teachers should understand the method.

The paper presents the development of the method step by step. It analyzes how the organization of the various methods came into use, and compares their advantages and disadvantages. The analyses show that the author's method adds several advantages to the present method without losing any beneficial aspects of the latter or any new math concepts. Two of the more significant advantages of the method are:

- *More effective grouping of basic addition facts, and systematic and consistent use of reasoning in their derivation.*
- *Use of a special class-room technique to improve the proficiency of a child in the application of basic addition facts.*

Tests conducted by the author on the basis of individual instruction showed that, by this method, a child learned the basic addition facts more easily and became proficient in their application in less time. With a few minutes of practice a day he achieved, in the first grade, a speed and skill in basic addition which otherwise he might not have achieved before the seventh or the eighth grade. Similar results could also be expected on the basis of class-room instruction.

1 + 7 = 8	6 + 5 = 11	3 + 2 = 5	7 + 3 = 10	4 + 1 = 5	8 + 3 = 11	9 + 9 = 18	5 + 1 = 6	8 + 6 = 14	
1 + 1 = 2	2 + 1 = 3	8 + 5 = 13	1 + 3 = 4	6 + 9 = 15	7 + 4 = 11	3 + 6 = 9	4 + 7 = 11	8 + 7 = 15	
2 + 3 = 5	3 + 8 = 11	6 + 4 = 10	1 + 6 = 7	8 + 8 = 16	7 + 8 = 15	3 + 5 = 8	3 + 9 = 12	5 + 7 = 12	3 + 7 = 10
9 + 3 = 12	1 + 9 = 10	2 + 4 = 6	4 + 5 = 9	9 + 8 = 17	2 + 7 = 9	1 + 2 = 3	4 + 5 = 9	2 + 5 = 7	
6 + 7 = 13	7 + 6 = 13	6 + 2 = 8	9 + 4 = 13	5 + 6 = 11	1 + 8 = 9	8 + 9 = 17	5 + 2 = 7	7 + 5 = 12	6 + 6 = 12
4 + 4 = 8	7 + 9 = 16	7 + 1 = 8	6 + 8 = 14	9 + 2 = 11	4 + 3 = 7	5 + 3 = 8	8 + 4 = 12		
3 + 1 = 4	2 + 9 = 11	9 + 7 = 16	3 + 4 = 7	4 + 8 = 12	2 + 8 = 10	4 + 9 = 13	8 + 1 = 9	4 + 2 = 6	
4 + 6 = 10	9 + 5 = 14	2 + 2 = 4	7 + 2 = 9	3 + 3 = 6	5 + 4 = 9	5 + 8 = 13	6 + 3 = 9	9 + 1 = 10	
8 + 2 = 10	5 + 5 = 10	7 + 7 = 14	6 + 1 = 7	1 + 5 = 6	1 + 4 = 5	9 + 6 = 15	5 + 2 = 7		

TABLE 1. ADDITION FACTS with NO PARTICULAR ORDER

1 + 1 = 2	2 + 1 = 3	3 + 1 = 4	4 + 1 = 5	5 + 1 = 6	6 + 1 = 7	7 + 1 = 8	8 + 1 = 9	9 + 1 = 10
1 + 2 = 3	2 + 2 = 4	3 + 2 = 5	4 + 2 = 6	5 + 2 = 7	6 + 2 = 8	7 + 2 = 9	8 + 2 = 10	9 + 2 = 11
1 + 3 = 4	2 + 3 = 5	3 + 3 = 6	4 + 3 = 7	5 + 3 = 8	6 + 3 = 9	7 + 3 = 10	8 + 3 = 11	9 + 3 = 12
1 + 4 = 5	2 + 4 = 6	3 + 4 = 7	4 + 4 = 8	5 + 4 = 9	6 + 4 = 10	7 + 4 = 11	8 + 4 = 12	9 + 4 = 13
1 + 5 = 6	2 + 5 = 7	3 + 5 = 8	4 + 5 = 9	5 + 5 = 10	6 + 5 = 11	7 + 5 = 12	8 + 5 = 13	9 + 5 = 14
1 + 6 = 7	2 + 6 = 8	3 + 6 = 9	4 + 6 = 10	5 + 6 = 11	6 + 6 = 12	7 + 6 = 13	8 + 6 = 14	9 + 6 = 15
1 + 7 = 8	2 + 7 = 9	3 + 7 = 10	4 + 7 = 11	5 + 7 = 12	6 + 7 = 13	7 + 7 = 14	8 + 7 = 15	9 + 7 = 16
1 + 8 = 9	2 + 8 = 10	3 + 8 = 11	4 + 8 = 12	5 + 8 = 13	6 + 8 = 14	7 + 8 = 15	8 + 8 = 16	9 + 8 = 17
1 + 9 = 10	2 + 9 = 11	3 + 9 = 12	4 + 9 = 13	5 + 9 = 14	6 + 9 = 15	7 + 9 = 16	8 + 9 = 17	9 + 9 = 18

TABLE 2. THE BASIC ADDITION BLOCK (ADDITION FACTS ORDERLY ARRANGED)

INTRODUCTION

To compute addition like the ones shown in Figure 1, a child must know the elementary or basic addition of two digits, such as $5 + 4 = 9$, $4 + 6 = 10$, etc. All such basic

546793	78649	741
+469831	+ 6943	+ 69

FIGURE 1. Examples of addition in practice

addition, or *basic addition facts*, as they are more commonly referred to, of two digits from 1 to 9 are shown, with no particular order, in Table 1. One could conceive of a time in the state of the art when addition facts were not arranged in order. Clearly then, man's first attempt was to arrange them in some order. Having known no other way, or having found no reason to arrange differently, he arranged them according to his very old, but not obsolete, natural sense of arranging objects, that is, in rows and columns in ascending order of the natural numbers. This arrangement is shown in Table 2.†

It is obvious that a child's proficiency in addition depends upon his proficiency in basic addition facts. By the author's method, a child could learn the basic addition facts more easily and could become proficient in their application in less time.

Adler [1] enlarges the basic addition block of Table 2 by including the zero addition facts in it. Referring to the enlarged addition table (page 112), he asks a child, at the completion of his third-grade year, to "know these 100 addition facts by heart." Upton and others [2] ask a child to "tell or write the answers as quickly as you can" to the addition problems of Table 1, but scrambled in a different manner (page 209). (An addition fact without the sum will be called an addition problem.)

In the present work, a method is developed in which we may expect the child to know the basic addition facts at the completion of his first-grade year, and in which we shall have reduced the size of the addition table and refer the child to a smaller table. We shall not expect the child to know all the addition facts by heart, but only some of them by heart, and to derive the rest from the ones he would know by heart. Further, we shall expect him to know only those facts by heart which are easier than the others. And for the others, we shall improve the present method, so that he can derive them more quickly.

DESCRIPTION OF METHODS

We see in the Introduction that the addition facts of Table 1, having no particular order, are arranged in Table 2, in ascending order of the natural numbers. After this is done, we must ask, "In which order do we introduce the addition facts to a child, since we cannot introduce them all at the same time?" For example, do we introduce them '"

† Table 2 has been labelled as the Basic Addition Block and will be referred to as such throughout the paper.

[1] "Mathematics—Grade 3," Irving Adler, Ph.D., Golden Press
[2] "Learning About Numbers," C. B. Upton, K. G. Fuller and G. H. McMeen, American Book Company

1 +1 2	2 +1 3	3 +1 4	4 +1 5	5 +1 6	6 +1 7	7 +1 8	8 +1 9	9 +1 10
1 +2 3	2 +2 4	3 +2 5	4 +2 6	5 +2 7	6 +2 8	7 +2 9	8 +2 10	9 +2 11
1 +3 4	2 +3 5	3 +3 6	4 +3 7	5 +3 8	6 +3 9	7 +3 10	8 +3 11	9 +3 12
1 +4 5	2 +4 6	3 +4 7	4 +4 8	5 +4 9	6 +4 10	7 +4 11	8 +4 12	9 +4 13
1 +5 6	2 +5 7	3 +5 8	4 +5 9	5 +5 10	6 +5 11	7 +5 12	8 +5 13	9 +5 14
1 +6 7	2 +6 8	3 +6 9	4 +6 10	5 +6 11	6 +6 12	7 +6 13	8 +6 14	9 +6 15
1 +7 8	2 +7 9	3 +7 10	4 +7 11	5 +7 12	6 +7 13	7 +7 14	8 +7 15	9 +7 16
1 +8 9	2 +8 10	3 +8 11	4 +8 12	5 +8 13	6 +8 14	7 +8 15	8 +8 16	9 +8 17
1 +9 10	2 +9 11	3 +9 12	4 +9 13	5 +9 14	6 +9 15	7 +9 16	8 +9 17	9 +9 18

TABLE 3. THE ADDITION TABLE ACCORDING TO THE OLD METHOD

1 →									
2 →									
3 →									
4 →									
5 →									
6 →									
7 →									
8 →									
9 →									

TABLE 3A. THE DIRECTION NET for THE OLD METHOD

the order $1 + 1, 2 + 1, 3 + 1, \ldots$, in the order $1 + 1, 1 + 2, 1 + 3, \ldots$, or in some other order? If we introduce them in the order $1 + 1, 2 + 1, 3 + 1, \ldots$, then a child travels the basic addition block along rows from left to right. If we introduce them in the order $1 + 1, 1 + 2, 1 + 3, \ldots$, then a child travels the basic addition block along columns from top to bottom, and so on. After we have decided how we shall introduce the addition facts to a child, the next question we must ask, "How do we teach him an individual addition fact?"

We shall review the various methods presented in this paper in the light of these two questions. The reader should, however, keep in mind that in this section of the paper we shall primarily be describing the various methods, but not analyzing or comparing them; analysis and comparison will be found in the next section.

THE OLD METHOD

In the old method, strictly speaking in one of the old methods, the basic addition facts were introduced to a child in the order $1 + 1, 2 + 1, 3 + 1, \ldots$ So, a child traveled the basic addition block along rows from left to right, top to bottom. This direction of travel is shown in Table 3. A reader may have observed that the basic addition block has not been altered; only a direction, shown by the direction net of Table 3A, has been superimposed on it. At one time in the old method, still used in some places, a child learned and performed the basic addition facts by finger-counting; that is, he counted fingers each time he computed the sum of *a* basic addition fact. For long addition, too, he depended upon finger-counting. The process of learning was very slow. (The term "long addition" has been used to differentiate between basic addition and addition of two numbers explicitly, each number containing more than one digit.)

The direction nets need some explanation. Light lines separate one addition fact from another, and heavy lines separate one group of addition facts from another. Arrows point the direction in which a child travels in a group, while the numbers by the arrows indicate how the groups are introduced to the child.

THE FLASH METHOD

In the so-called flash method, addition facts of the basic addition block were flashed before a child, using flash cards. This was done with the hope that after a time, when addition facts were shown to him without the sum, the sum would appear in his memory. This process facilitated drilling and improved a child's speed of learning. But the child still depended upon his memory, for, the sum must be retrieved from his memory. He still followed rows from left to right; that is, he still followed the direction of the old method. It may be mentioned here that the flash method is *not* a basic method. It is a system of training in which flash cards are employed to improve the speed and skill of a child in acquiring and in applying the basic addition facts. *The system does not have a direction of its own. It uses* *the direction of the method which is being employed at a particular time.*

Some of the flash cards that are available are flash cards No. 7020, published by the Milton Bradley Company, flash cards No. 96700, published by the McGraw-Hill Book Company, and flash cards No. 4570:39, published by the Western Publishing Company. All of these flash cards use the direction of travel of the old method. Figure 2, which is a summary card of the Milton Bradley flash cards No. 7020, shows this direction of travel, for example.

THE CONSTANT-SUM METHOD

The order in which the basic addition facts are introduced to a child in the method currently in use in the United States is shown in Table 4. The method currently in use in the United States is referred to as the *constant-sum* method. The reason for the name is discussed shortly. It is seen that, in this method, a child travels the basic addition block diagonally. The reader may have observed again that the basic addition block of Table 2 has not been altered; only a new direction, shown by the direction net of Table 4A, has been superimposed on it. This direction of travel has been used by authors of first and second grade arithmetic books in various modifications. For the benefit of those who may not be familiar with the present system of elementary education, the following books may be cited as examples.

1. "Elementary Mathematics," Second Edition, Grade 1, Harcourt, Brace & World, Inc
2. "Arithmetic Workshop," Second Edition, Book 1, American Book Company
3. "Elementary School Mathematics," Second Edition, Book 1, Addison-Wesley Publishing Company

It is seen that the sums of the addition facts of a step in this method remain constant for all the facts of the same step. For this reason, this method has been referred to as the constant-sum method.

To make the direction of travel of the constant-sum method conform with our traditional sense of arrangement of objects, its addition facts have been rearranged from Table 4 into Table 5. The first nine steps of Table 5 are taught to a child chiefly through his basic training in counting, concepts and understanding of numbers. The remaining six steps are taught with the help of the first nine steps, and by use of reasoning or derivation. In this method, a child must complete the addition facts of the first nine steps before he can use reasoning to derive *each* fact of the remaining six steps. In a way therefore we may regard the first nine steps of Table 5 as the fundamental steps of addition and the remaining six steps of Table 5 as the additional steps of addition, in the constant-sum method. The reader may have observed that *the flash system of training cannot be used in the constant-sum method,* since the sum is constant for all the facts of the same step. An elaboration of this point is found in a later section of this paper.

BASIC ADDITION FACTS

$1+1=\boxed{2}$	$1+2=\boxed{3}$	$4+3=\boxed{7}$	$7+4=\boxed{11}$	$1+6=\boxed{7}$	$4+7=\boxed{11}$	$7+8=\boxed{15}$
$2+1=\boxed{3}$	$2+2=\boxed{4}$	$5+3=\boxed{8}$	$8+4=\boxed{12}$	$2+6=\boxed{8}$	$5+7=\boxed{12}$	$8+8=\boxed{16}$
$3+1=\boxed{4}$	$3+2=\boxed{5}$	$6+3=\boxed{9}$	$9+4=\boxed{13}$	$3+6=\boxed{9}$	$6+7=\boxed{13}$	$9+8=\boxed{17}$
$4+1=\boxed{5}$	$4+2=\boxed{6}$	$7+3=\boxed{10}$	$1+5=\boxed{6}$	$4+6=\boxed{10}$	$7+7=\boxed{14}$	$1+9=\boxed{10}$
$5+1=\boxed{6}$	$5+2=\boxed{7}$	$8+3=\boxed{11}$	$2+5=\boxed{7}$	$5+6=\boxed{11}$	$8+7=\boxed{15}$	$2+9=\boxed{11}$
$6+1=\boxed{7}$	$6+2=\boxed{8}$	$9+3=\boxed{12}$	$3+5=\boxed{8}$	$6+6=\boxed{12}$	$9+7=\boxed{16}$	$3+9=\boxed{12}$
$7+1=\boxed{8}$	$7+2=\boxed{9}$	$1+4=\boxed{5}$	$4+5=\boxed{9}$	$7+6=\boxed{13}$	$1+8=\boxed{9}$	$4+9=\boxed{13}$
$8+1=\boxed{9}$	$8+2=\boxed{10}$	$2+4=\boxed{6}$	$5+5=\boxed{10}$	$8+6=\boxed{14}$	$2+8=\boxed{10}$	$5+9=\boxed{14}$
$9+1=\boxed{10}$	$9+2=\boxed{11}$	$3+4=\boxed{7}$	$6+5=\boxed{11}$	$9+6=\boxed{15}$	$3+8=\boxed{11}$	$6+9=\boxed{15}$
	$1+3=\boxed{4}$	$4+4=\boxed{8}$	$7+5=\boxed{12}$	$1+7=\boxed{8}$	$4+8=\boxed{12}$	$7+9=\boxed{16}$
	$2+3=\boxed{5}$	$5+4=\boxed{9}$	$8+5=\boxed{13}$	$2+7=\boxed{9}$	$5+8=\boxed{13}$	$8+9=\boxed{17}$
	$3+3=\boxed{6}$	$6+4=\boxed{10}$	$9+5=\boxed{14}$	$3+7=\boxed{10}$	$6+8=\boxed{14}$	$9+9=\boxed{18}$

see the other side No. 7020

FIGURE 2. The summary card of Milton Bradley Flash Cards (By permission)

1 +1 2	2 +1 3	3 +1 4	4 +1 5	5 +1 6	6 +1 7	7 +1 8	8 +1 9	9 +1 10
1 +2 3	2 +2 4	3 +2 5	4 +2 6	5 +2 7	6 +2 8	7 +2 9	8 +2 10	9 +2 11
1 +3 4	2 +3 5	3 +3 6	4 +3 7	5 +3 8	6 +3 9	7 +3 10	8 +3 11	9 +3 12
1 +4 5	2 +4 6	3 +4 7	4 +4 8	5 +4 9	6 +4 10	7 +4 11	8 +4 12	9 +4 13
1 +5 6	2 +5 7	3 +5 8	4 +5 9	5 +5 10	6 +5 11	7 +5 12	8 +5 13	9 +5 14
1 +6 7	2 +6 8	3 +6 9	4 +6 10	5 +6 11	6 +6 12	7 +6 13	8 +6 14	9 +6 15
1 +7 8	2 +7 9	3 +7 10	4 +7 11	5 +7 12	6 +7 13	7 +7 14	8 +7 15	9 +7 16
1 +8 9	2 +8 10	3 +8 11	4 +8 12	5 +8 13	6 +8 14	7 +8 15	8 +8 16	9 +8 17
1 +9 10	2 +9 11	3 +9 12	4 +9 13	5 +9 14	6 +9 15	7 +9 16	8 +9 17	9 +9 18

TABLE 4. THE BASIC ADDITION BLOCK arranged for THE CONSTANT-SUM METHOD

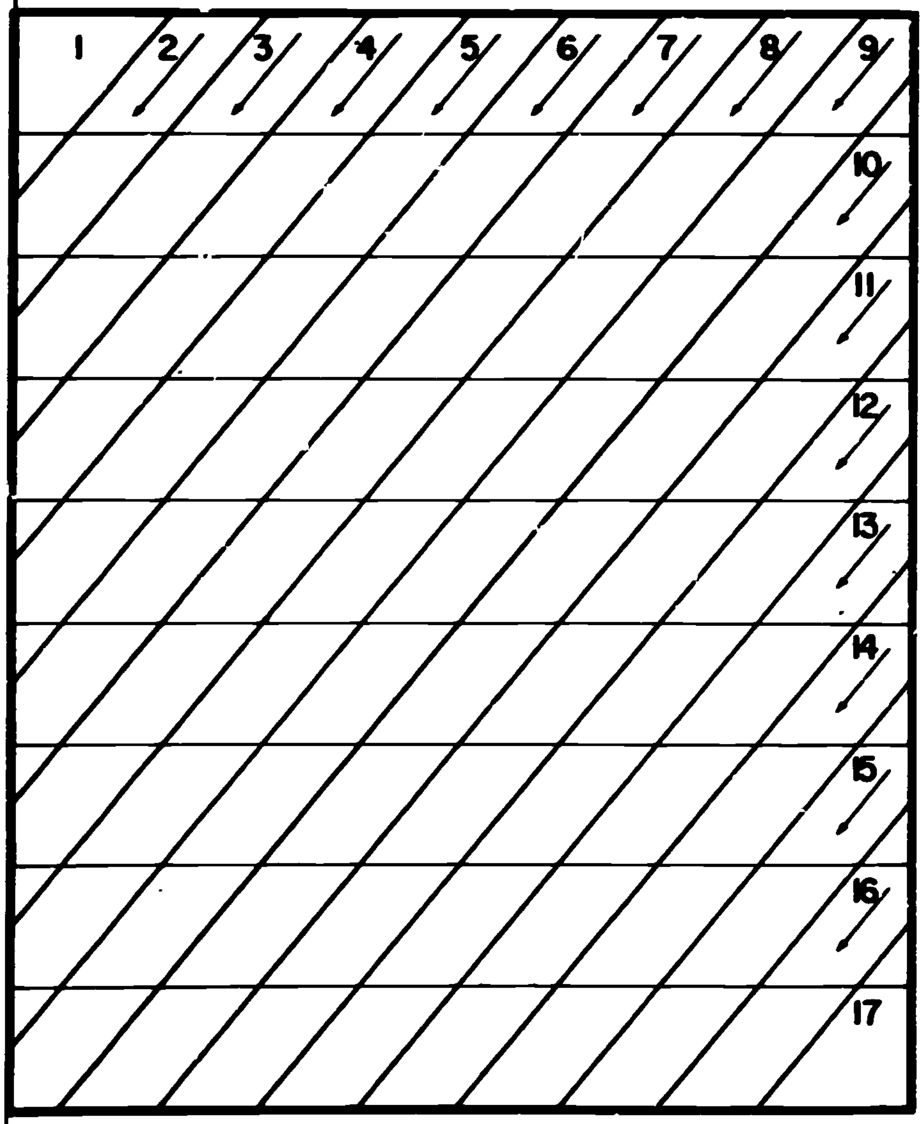

TABLE 4A. THE DIRECTION NET for THE CONSTANT-SUM METHOD

STEPS									
1	1 +1 2								
2	2 +1 3	1 +2 3							
3	3 +1 4	2 +2 4	1 +3 4						
4	4 +1 5	3 +2 5	2 +3 5	1 +4 5					
5	5 +1 6	4 +2 6	3 +3 6	2 +4 6	1 +5 6				
6	6 +1 7	5 +2 7	4 +3 7	3 +4 7	2 +5 7	1 +6 7			
7	7 +1 8	6 +2 8	5 +3 8	4 +4 8	3 +5 8	2 +6 8	1 +7 8		
8	8 +1 9	7 +2 9	6 +3 9	5 +4 9	4 +5 9	3 +6 9	2 +7 9	1 +8 9	
9	9 +1 10	8 +2 10	7 +3 10	6 +4 10	5 +5 10	4 +6 10	3 +7 10	2 +8 10	1 +9 10
10	9 +2 11	8 +3 11	7 +4 11	6 +5 11	5 +6 11	4 +7 11	3 +8 11	2 +9 11	
11	9 +3 12	8 +4 12	7 +5 12	6 +6 12	5 +7 12	4 +8 12	3 +9 12		
12	9 +4 13	8 +5 13	7 +6 13	6 +7 13	5 +8 13	4 +9 13			
13	9 +5 14	8 +6 14	7 +7 14	6 +8 14	5 +9 14				
14	9 +6 15	8 +7 15	7 +8 15	6 +9 15					
15	9 +7 16	8 +8 16	7 +9 16						
16	9 +8 17	8 +9 17							
17	9 +9 18								

TABLE 5. THE ADDITION TABLE by THE CONSTANT-SUM METHOD

1 + 1 = 2	2 + 1 = 3	3 + 1 = 4	4 + 1 = 5	5 + 1 = 6	6 + 1 = 7	7 + 1 = 8	8 + 1 = 9	9 + 1 = 10
1 + 2 = 3	2 + 2 = 4	3 + 2 = 5	4 + 2 = 6	5 + 2 = 7	6 + 2 = 8	7 + 2 = 9	8 + 2 = 10	9 + 2 = 11
1 + 3 = 4	2 + 3 = 5	3 + 3 = 6	4 + 3 = 7	5 + 3 = 8	6 + 3 = 9	7 + 3 = 10	8 + 3 = 11	9 + 3 = 12
1 + 4 = 5	2 + 4 = 6	3 + 4 = 7	4 + 4 = 8	5 + 4 = 9	6 + 4 = 10	7 + 4 = 11	8 + 4 = 12	9 + 4 = 13
1 + 5 = 6	2 + 5 = 7	3 + 5 = 8	4 + 5 = 9	5 + 5 = 10	6 + 5 = 11	7 + 5 = 12	8 + 5 = 13	9 + 5 = 14
1 + 6 = 7	2 + 6 = 8	3 + 6 = 9	4 + 6 = 10	5 + 6 = 11	6 + 6 = 12	7 + 6 = 13	8 + 6 = 14	9 + 6 = 15
1 + 7 = 8	2 + 7 = 9	3 + 7 = 10	4 + 7 = 11	5 + 7 = 12	6 + 7 = 13	7 + 7 = 14	8 + 7 = 15	9 + 7 = 16
1 + 8 = 9	2 + 8 = 10	3 + 8 = 11	4 + 8 = 12	5 + 8 = 13	6 + 8 = 14	7 + 8 = 15	8 + 8 = 16	9 + 8 = 17
1 + 9 = 10	2 + 9 = 11	3 + 9 = 12	4 + 9 = 13	5 + 9 = 14	6 + 9 = 15	7 + 9 = 16	8 + 9 = 17	9 + 9 = 18

TABLE 6. THE BASIC ADDITION BLOCK arranged for THE CONSISTENT-LOGIC METHOD

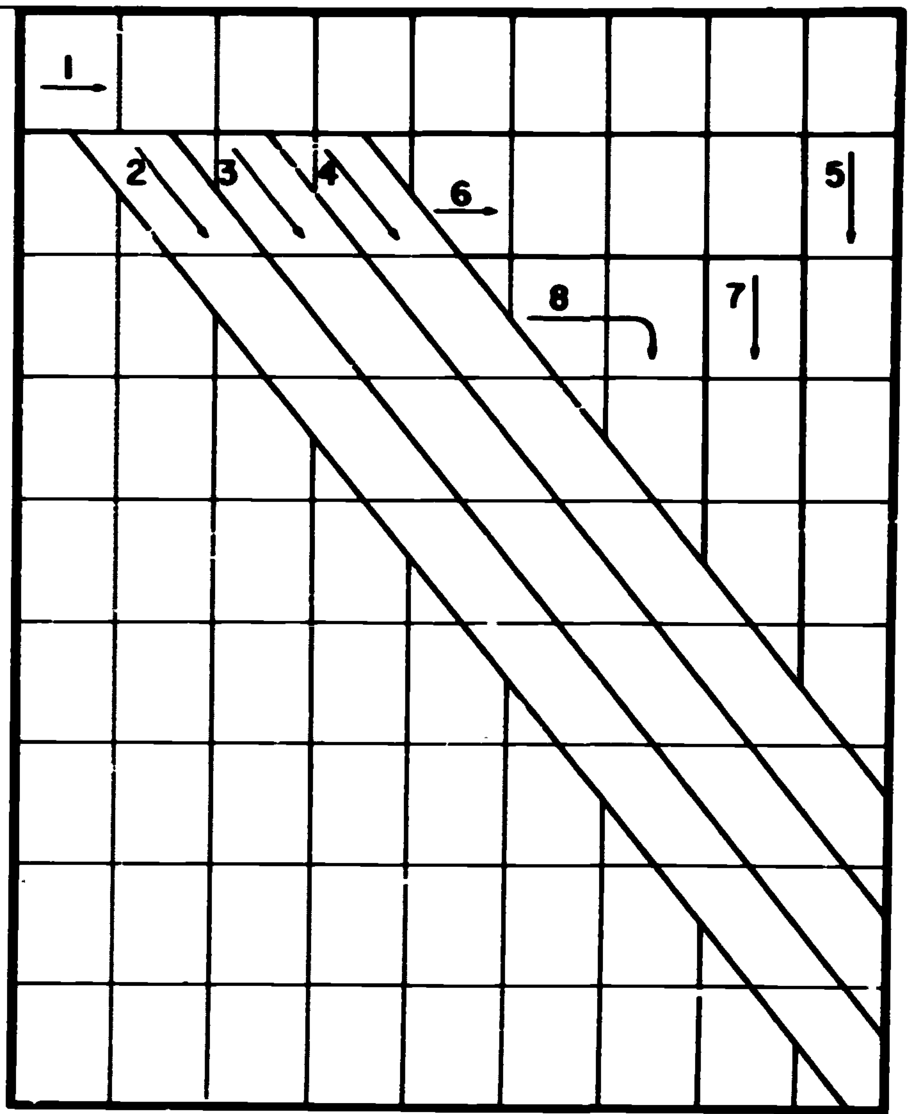

TABLE 6A. THE DIRECTION NET for THE CONSISTENT-LOGIC METHOD

THE CONSISTENT-LOGIC METHOD

The method presented in this paper will be referred to as the *consistent-logic* method. The reason for the name follows shortly. The order in which the basic addition facts will be introduced to a child in the consistent-logic method is shown in Table 6. Again, it may be observed that the basic addition block has not been altered; only a new direction, shown by the direction net of Table 6A, has been superimposed on it. In this new direction, a child travels the basic addition block as follows:

> He travels the first row first from left to right, as he did in the old method. He then travels the diagonals, starting with $2 + 2$, $3 + 2$, $4 + 2$, respectively. Next, he travels the column starting with $9 + 2$, the row starting with $5 + 2$, the column starting with $8 + 3$, and finally, the small triangle starting with $6 + 3$.

Arranged to conform with our traditional sense of arrangement of objects, the addition facts of the consistent-logic method are shown from Table 6 into Table 7. It may be observed that we did not include the addition facts of the lower triangle of Table 6 in Table 7. We shall see later why we did not do so.

The first two steps of Table 7 are fundamental steps, and we may refer to the remaining six steps as the additional steps of addition, defined by the same criteria of definition we employed for the fundamental and additional steps of addition in the constant-sum method. As in the constant-sum method, the fundamental steps of the consistent-logic method are taught to a child through his basic training in counting, concepts, and understanding of numbers, and the additional steps, with the help of the fundamental steps, and by use of reasoning or derivation.

The use of reasoning for derivation of addition facts is *not a new concept in the consistent-logic method.* Again, for the benefit of those who may not be familiar with the new math as applied to elementary grades, it may be pointed out that children are now trained in the use of reasoning, whenever applicable, to acquire basic addition facts. For example, a child is now trained to learn the fact $9 + 4 = 13$, as follows.†

> He first learns the fact $10 + 3 = 13$, through his basic training in counting by tens and ones (10 and 1 are 11, 10 and 2 are 12, . . . are counting by tens and ones). Then he is trained to reason, "take 1 from 4 and put it (1) to 9, we have 10 and 3 are 13" (Figure 3).

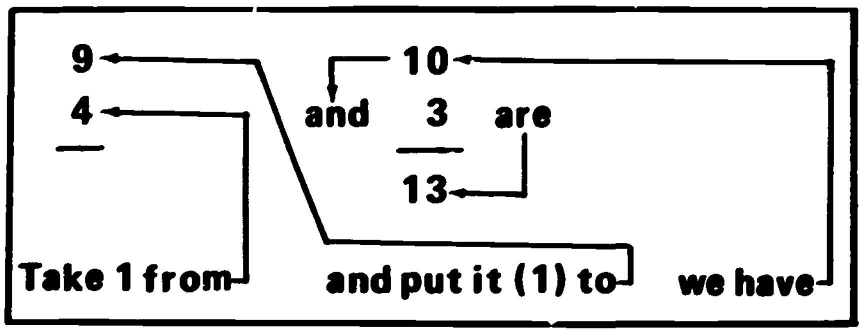

FIGURE 3. An example of use of reasoning for derivation of addition facts

† See page 245 of "Elementary School Mathematics," cited previously.

STEPS									
1	1 +1 2	2 +1 3	3 +1 4	4 +1 5	5 +1 6	6 +1 7	7 +1 8	8 +1 9	9 +1 10
2	2 +2 4	3 +3 6	4 +4 8	5 +5 10	6 +6 12	7 +7 14	8 +8 16	9 +9 18	
3	3 +2 5	4 +3 7	5 +4 9	6 +5 11	7 +6 13	8 +7 15	9 +8 17		
4	4 +2 6	5 +3 8	6 +4 10	7 +5 12	8 +6 14	9 +7 16			
5	9 +2 11	9 +3 12	9 +4 13	9 +5 14	9 +6 15				
6	5 +2 7	6 +2 8	7 +2 9	8 +2 10					
7	8 +3 11	8 +4 12	8 +5 13						
8	6 +3 9	7 +3 10	7 +4 11						

TABLE 7. THE ADDITION TABLE
by THE CONSISTENT-LOGIC METHOD

He is demonstrated the reasoning with some visual aids or objects, and he is exposed to the use of reasoning only after he has received and acquired a reasonable training in concepts and understanding of numbers. Although the use of reasoning is not a new concept in the consistent-logic method, we shall see later that reasoning is systematic and consistent in this method. For this reason, we have referred to this method as the consistent-logic method. And as to the use of flash system of training, we see that it can still be used in the consistent-logic method.

Reasoning of the kind discussed here with the help of Figure 3 cannot be very easily explained without a personal demonstration. For readers other than teachers, it may not have been very clear. Since, however, this reasoning is an important part for the discussion to follow, it would be appropriate for a reader to pause here for a moment to review the reasoning, and to grasp it fairly well.

SUMMARY OF METHODS

One could conceive of a time when the addition facts were not organized in order. Man's first attempt was to organize them in some order, and he did so, as shown in Table 2. Later, different methods were developed for teaching basic addition, depending upon how the addition facts were introduced to a child. In one of the old methods, the addition facts were introduced to a child in the order as shown in Table 3. The order in which they are introduced in the constant-sum method and in the consistent-logic method are shown in Table 5 and Table 7, respectively. The method currently in use in the United States has been called the constant-sum method, and the method developed by the author is called consistent-logic method.

ADOPTION OF THE CONSISTENT-LOGIC METHOD

As stated in the Abstract, the consistent-logic method can be used in the arithmetic books for first and second grades. The adoption of the method will not require any change in the arithmetic books for upper grades, unless it is desired to upgrade the curriculum and the standard of instruction. The adoption of the method also will not require any change *in the preparatory training* of a child from kindergarten to that stage of the first grade when he first begins his formal training in basic addition facts, nor will it require any change *in the training for application* of basic addition facts, after he has acquired them. All materials, methods, or tools used in the preparatory training and in the training for application will remain unaffected, as well as the ways they are used. The change will be made *during the actual period of training* in basic addition facts, and only *in the order of presenting them* according to the consistent-logic method.

In view of the preceding discussion, which is pictorially illustrated in Figure 4, we may conclude that the consistent-logic method will not require new or special training for teachers, nor will it require complete rewriting of first and second grade arithmetic books. It will be sufficient for teachers to have an understanding of the method, and authors may revise their books, to write only those portions

PERIOD OF PREPARATORY BASIC TRAINING	ACTUAL PERIOD OF TRAINING IN THE BASIC ADDITION FACTS	PERIOD OF TRAINING IN THE APPLICATION OF BASIC ADDITION FACTS
NO CHANGE	CHANGE	NO CHANGE

FIGURE 4. Changes relative to periods of training
when the consistent-logic method is used

which were devoted to the training in basic addition facts, according to the constant-sum method.

A second paper is being prepared to systematize the consistent-logic method further and to outline broad guidelines for revision of existing books and for writing of new books. The paper will also list the different ways in which the consistent-logic method can be presented in teaching materials and will describe the flexibility with which it can be used in various situations.

$$\begin{array}{r} 3 \\ +2 \\ \hline \end{array} \qquad \begin{array}{r} 2 \\ +2 \\ \hline 4 \\ +1 \\ \hline 5 \end{array} \qquad\qquad \begin{array}{r} 4 \\ +3 \\ \hline \end{array} \qquad \begin{array}{r} 3 \\ +3 \\ \hline 6 \\ +1 \\ \hline 7 \end{array}$$

FIGURE 5. First example of use of same logic and fundamental steps for derivation of addition facts by the consistent-logic method

COMPARISON AND ANALYSIS OF METHODS

Why Comparison And Analysis?

The consistent-logic method was developed through experimentation with children for more than one year. After it was developed, the method was tested on the basis of individual instruction. Tests showed that, by the consistent-logic method, a child learned the basic addition facts more easily and quickly. Comparison and analysis are presented to explain the results of tests and to give some insight into the various methods. Insight into the various methods may point out some of the reasons that could have contributed to the improvement of the consistent-logic method. Other reasons for analyses are found in the Conclusion. It may be more interesting to a reader to have his own comparison and analysis before proceeding to those that follow.

Fundamental Facts Reduced In Number

In the old method, all 81 basic addition facts were fundamentals to a child, since he learned each of them by finger-counting or by rote memorization. In the constant-sum method, there are 9 fundamental steps, with 45 basic addition facts. In the consistent-logic method, there are two fundamental steps, with 17 basic addition facts. The consistent-logic method has, therefore, the least number of fundamentals.†

The fact that the consistent-logic method has the least number of fundamentals remains true also after we exclude the inverse addition facts from Table 5 of the constant-sum method much as we have excluded them from Table 7 of the consistent-logic method. (An inverse addition fact here is defined to be one in which two numerals to be added are reversed, for example, $1 + 7 = 8$ is the inverse addition fact of $7 + 1 = 8$.) However, we must not overlook the fact that the inclusion of the inverse addition facts in Table 5 was a necessary part of the constant-sum method, whereas their exclusion from Table 7 is not an exclusion of a necessary part of the consistent-logic method.

Fundamental Facts Easier To Learn And Remember

Tests showed that the fundamental facts of the consistent-logic method were easier to learn and remember than those of the constant-sum method. One could intuitively conclude so by inspecting them. However, some explanation is provided to support test results.

The facts $1 + 1, 2 + 1, 3 + 1, \ldots$ are present in both sets of fundamentals. They could not alter the ease with which a child learns the fundamental facts in either method. Fortunately, therefore, we need compare the remaining fundamental facts. We shall do so with the help of examples. For example, we take the fact $6 + 6 = 12$ from step 2 of the consistent-logic method, and see how a child learns this fact. We see that a child could learn this fact directly after his training in counting,‡ as a natural second step. The point being brought out is that there is no additional fact in between a child's training in counting and his learning of the fact $6 + 6 = 12$. But such is not the case for all the fundamental facts of the constant-sum method. For example, a child must know the fact $7 + 2 = 9$ before he could learn the fact $7 + 3 = 10$ of step 9 of the constant-sum method.

We also see that the facts of step 2 of the consistent-logic method are symmetrical. They are symmetrical in the sense that the two numerals to be added are same about the plus (+) sign of the problem. This symmetry could have helped a child learn these facts more easily, and remember them longer (or permanently) once they are learned. The fundamental facts of the constant-sum method are not all symmetrical.

More Effective Grouping

In the old method, each fact of the basic addition block was new to a child, because there were no fundamental steps that could be advantageously used to derive the rest. This serious drawback of the old method, of having no fundamental steps that could be advantageously used to derive the rest, was overcome in both the constant-sum method and in the consistent-logic method. This was done by *special grouping* of basic addition facts of Table 2 according to certain rules or concepts. (This concept of special grouping has been known as the concept of sets and subsets in the new math.)

Grouping in the constant-sum method is mechanical; that is, addition facts having the same sum are mechanically grouped together. For example, addition facts having the sum 13 are grouped in step 12 of Table 5, and addition facts having the sum 16 are grouped in step 15.

Grouping in the consistent-logic method is logical; that is, addition facts whose sums are derived by the use of

† Fundamental facts were defined earlier.

‡ In the new math a child is trained to count upward to 20 by twos.

same logic and *same* fundamental steps are grouped together. For example, all the addition facts of step 3 of Table 7 use the logic of 1 more † and the fundamental step 2. This is explained with the help of Figure 5. For the first fact, the logic is—"2 and 2 are 4 and 1 more are 5." For the second fact, the logic is—"3 and 3 are 6 and 1 more are 7." All the addition facts of step 4 use the logic of 1 more and 1 less with the fundamental step 2. This is explained in Figure 6. For the first fact, the logic is—"take 1 from 4 and put it to 2, we have 3 and 3 are 6." For the second fact, the logic is—"take 1 from 5 and put it to 3, we have 4 and 4 are 8."

$$\begin{array}{rr} 4 & 3 \\ +2 & +3 \\ \hline & 6 \end{array} \qquad \begin{array}{rr} 5 & 4 \\ +3 & +4 \\ \hline & 8 \end{array}$$

FIGURE 6. Second example of use of same logic and fundamental steps for derivation of addition facts by the consistent-logic method

Logic Systematic And Consistent

We have seen that grouping has reduced the number of fundamentals both in the constant-sum method and in the consistent-logic method from those present in the old method. Grouping has also made it possible to use logic (reasoning) in the derivation of additional steps of addition in these two methods. Logic is systematic and consistent in the consistent-logic method, but not so in the constant-sum method. This is shown below.

In the constant-sum method, logic changes from one addition fact to another in the same step. For example, the first fact of step 12 of Table 5 uses the logic, "take *1* from 4 and put it to 9, we have 10 and 3 are 13," as shown in Figure 7. But the second fact of the same step uses the logic, "take *2* from 5 and put it to 8," etc., which is a different logic. In the second case, we are taking away *2* instead of *1*. The third addition fact uses another logic yet. But in the consistent-logic method, logic does not change from one addition fact to another of the same step, as we saw in the description under "More Effective Grouping."

$$\begin{array}{rr} 9 & 10 \\ +4 & +3 \\ \hline & 13 \end{array} \qquad \begin{array}{rr} 8 & 10 \\ +5 & +3 \\ \hline & 13 \end{array}$$

FIGURE 7. An example of use of different logic and fundamental steps for derivation of addition facts by the constant-sum method

Because logic changes from one addition fact to another in the same step of the constant-sum method, each addition fact of the constant-sum method seems new to a learning child, a drawback which we wanted to eliminate from the old method. So, we can say that the constant-sum method has not used grouping very effectively, while, on the other hand, we see that the consistent-logic method has.

Continuing our analysis, we can also say that this drawback of the old method, still existing in the constant-sum method, has been eliminated from the consistent-logic method by logical grouping of its basic addition facts. Fortunately enough, grouping of the consistent-logic method still does or is capable of doing what grouping of the constant-sum method does. In addition, it uses logic consistently for all the addition facts of the same step. Because logic does not change from one addition fact to another in the same step of the consistent-logic method, we may expect a child to remember which logic to use in the subsequent fact once he is given the logic in the first fact, especially after some practice.

The Special Class-Room Technique

Earlier in the discussion, we stated that the flash method of training could not be employed in the constant-sum method. This is elaborated here. If we take a set of flash cards containing the addition facts of one of the groups of the constant-sum method, we see that the sum remains the same for all the facts of the group. Since the sum remains the same, a child will know the answers to all the problems once he knows the answer to one problem. Therefore, in the constant-sum method, a child could not be drilled in the addition facts of a group which was being taught at the time. This may explain why there are no flash cards, in the market, using the grouping of the constant-sum method.

In line with the use of flash cards, some class-room teaching cards have been conceived. These are 9×7 inch cards for the teacher as shown in Figure 8, and a set of 5×3 inch cards for the students as shown in Figure 9. These class-room teaching cards may be used to drill students in basic addition by the consistent-logic method. Drilling may begin after an understanding of logic has been established, and, *simultaneously* may be conducted as follows.

The teacher will take a set of 9×7 inch cards containing the addition problems of a step of the consistent-logic method. Each student will be given a set of 5×3 inch cards consisting of the sums of the given addition problems, plus an extra card not containing an answer. The teacher will face the class and show one problem to the class. The class will search for the card that has the correct answer and will show it to the teacher. The teacher can easily check all the answers at the same time when the students will hold up their answer cards.

No Loss Of New Math Concepts

One of the important advantages of the new math, as used in the constant-sum method, is that it trains a child in the practice of different arithmetic operations with the same three numerals of an addition fact. These different arithmetic operations with the same three numerals of an

† An example of the simple logic of 1 more is—"5 and 1 more are 6."

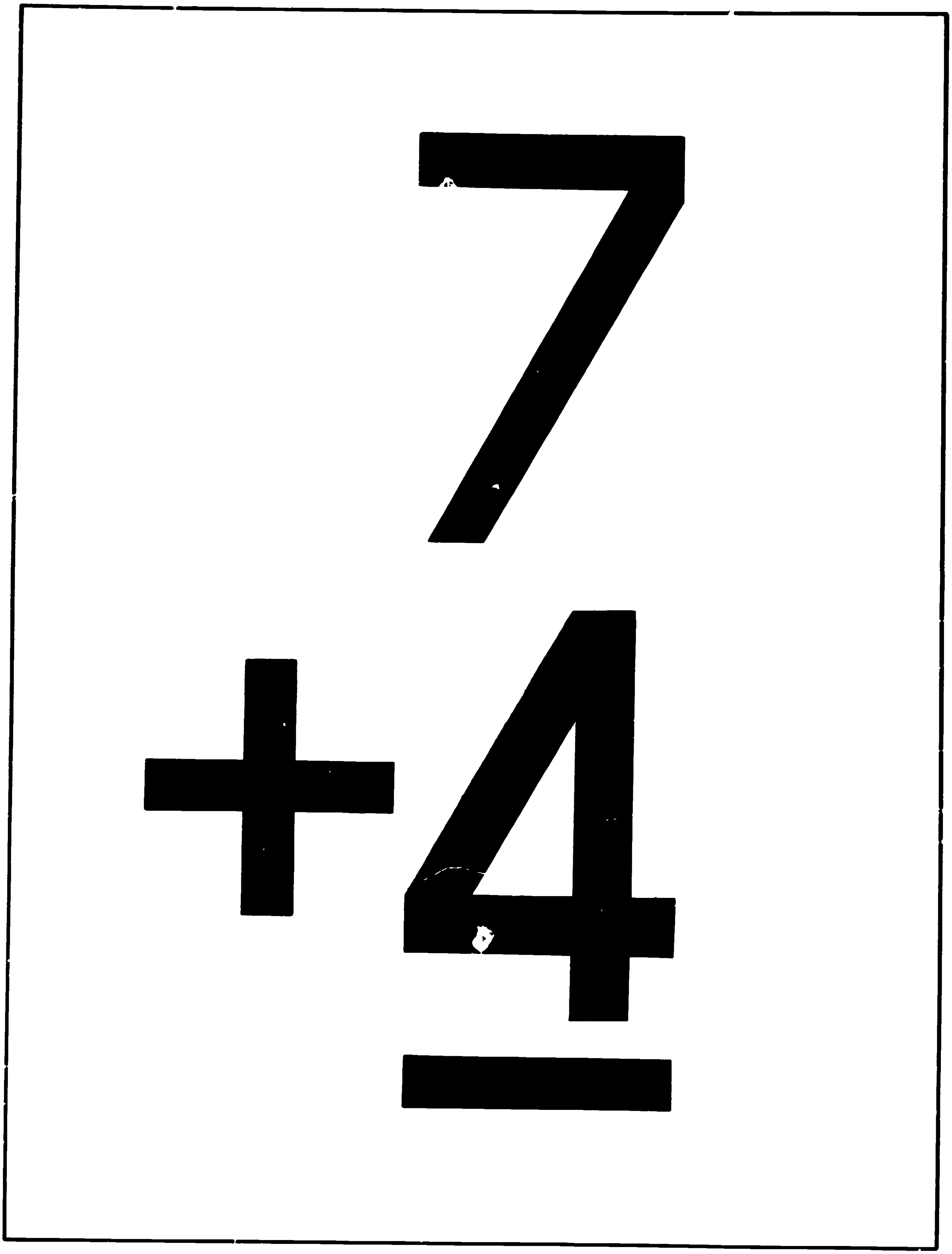

FIGURE 8. A teachers' teaching card

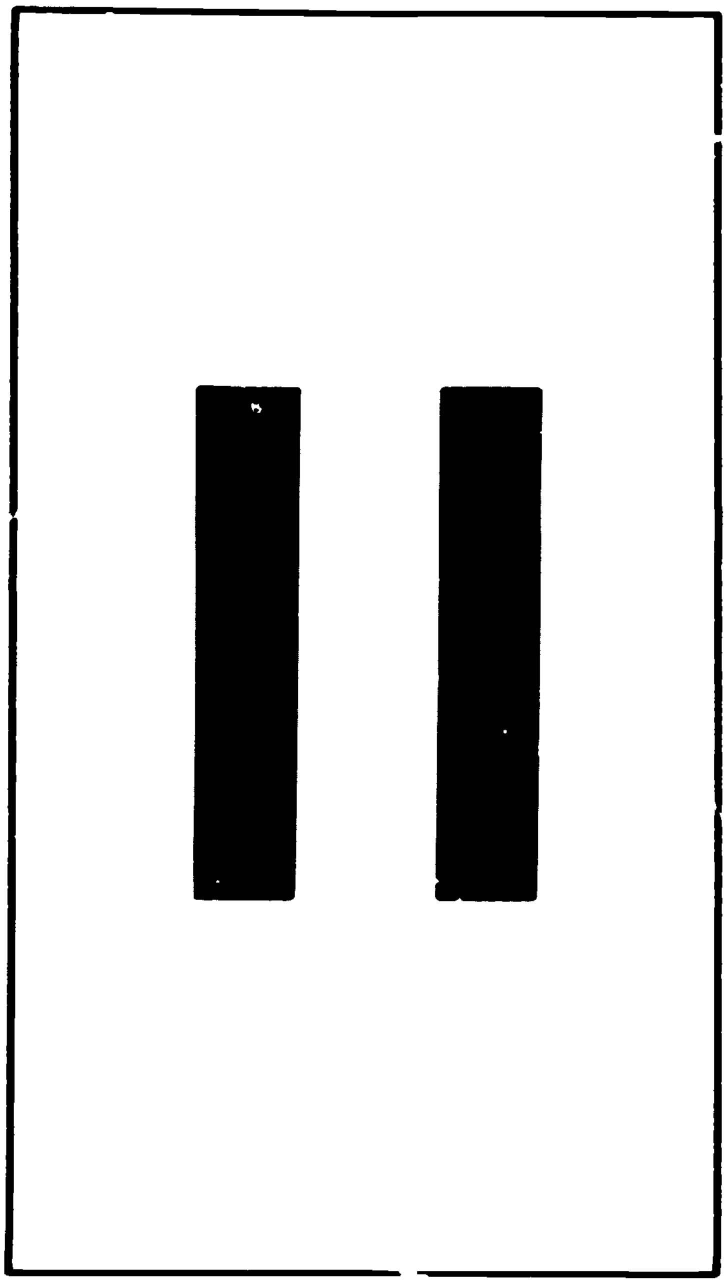

FIGURE 9. A students' learning card

9+3=12	12-9=3	N+3=12	12-N=3
3+9=12	12-3=9	9+N=12	12-N=9

FIGURE 10. Different arithmetic operations with same three numerals of an addition fact

addition fact are shown in Figure 10 for the fact $9 + 3 = 12$. Practice in performing these operations has resulted in improving a child's proficiency in application of *a basic addition fact,* and at the same time, a child learns subtractions and equations involving numbers up to 20. In the consistent-logic method also, a child can perform these operations. If we, further, considered other beneficial aspects of the constant-sum method or other new math concepts, we would see that the consistent-logic method has lost none of them. However, tests showed that the same results could be expected in these operations with less work on the child if he were trained by the consistent-logic method. The results of tests are explained as follows.

In the consistent-logic method, a basic addition fact such as $9 + 3 = 12$ is taught with emphasis on it as fundamental information. The remaining seven operations are taught as examples of how to apply fundamental information in practical problems. It was found that if a child acquired high proficiency in *a basic addition fact* in less time, he could learn the remaining seven derivatives of *it* very easily with a slight adjustment in his thinking. Since we found that, by the consistent-logic method, a child acquired high proficiency in less time, we could, at our option, take some advantage of the consistent-logic method in reducing the work-load which otherwise would be placed upon the child if he practiced basic addition facts and their derivatives with equal emphasis on both. For this reason, Table 7 does not contain the addition facts of the lower triangle of Table 6, namely, the inverse addition facts, nor does Table 6 include them in the direction assignment.

SUMMARY OF COMPARISON AND ANALYSIS

In the old method, all 81 basic addition facts were fundamentals. In the constant-sum method, this number was reduced to 45, and in the consistent-logic method, this number was further reduced to 17. In the consistent-logic method, not only are the fundamentals minimum in number, but they are also easier to learn and remember.

In the old method, each addition fact of the basic addition block was new to a child. To overcome this drawback, both the constant-sum method and the consistent-logic method group basic addition facts, with the latter method grouping them more effectively. Both the constant-sum method and the consistent-logic method use reasoning in the derivation of addition facts, with the latter method using it systematically and consistently.

The flash method of training can be employed in the old method and in the consistent-logic method, but it cannot be employed in the constant-sum method. Lastly, no new math concepts or beneficial aspects of the constant-sum method have been lost in the consistent-logic method.

CONCLUSION

The consistent-logic method is not a new method. It has some new features or concepts. In all other respects, it can be used *similarly* as the existing methods. Stated differently, the added features of the consistent-logic method can be incorporated in the existing methods. Some readers may see enough strength in the analyses for the conclusion that the consistent-logic method should help a child learn basic addition more easily and quickly; others may like to see the analyses verified by formal tests on the basis of group instruction.†

† As the author's tests were neither formal nor based on group instruction.

DISCUSSION

BY PROFESSOR SCHINDLER: †

I want to commend Mr. Aziz for his presentation of addition facts. Especially, I want to commend Table 7 as a way of showing the goal to be achieved in teaching addition and subtraction. Some teachers may not clearly see the goal to be achieved. As for pupils, the comment of a third grader is significant. When a teacher showed pupils a table of the facts, one pupil remarked: "If there are only that many, we can learn them. I thought that there were thousands of them." Mr. Aziz's table make the task look like one that can be accomplished, and that may be a great fountain of motivation.

Mr. Aziz does not have a total program. It seems to me that teachers and supervisors will want to see a total program into which his plan is integrated. They will want to know what the learning activities with children should be before they are ready for step 1.

In some respects there is very little that is absolutely new in a field which has been worked as much as elementary school arithmetic. For example, generalization (understanding and logic) have been emphasized as approaches to mastery of facts. Grossnickle and Brueckner have encouraged teachers to help pupils develop grouping of facts which resemble Aziz's steps in Table 7. Their groups were: the zero facts, the ones (his step 1), the doubles and near doubles, and the couple of others. As far as I know, a concise arrangement such as Mr. Aziz has in Table 7 was not used before. The method has merits, and it will be a contribution if a total program is developed satisfactorily.

THE AUTHOR'S CLOSURE

Professor Schindler has viewed Table 7 as a goal to be achieved. In the second paper on the method, Table 7 has been further divided into two separate tables (or goals). They are Table 8 and Table 9 as shown. The development and usefulness of these tables have been fully described in that paper.

Table 8 contains only the fundamental steps, and Table 9, the additional steps of addition. Steps of Table 8, in the second development, may be more significantly classified as fundamental steps with the following criteria.

1. A child naturally goes to these steps after his training in counting.
2. These steps are easier than the others.
3. And once learned, a child does not easily forget them, they become a part of his permanent knowledge.

As for Table 9, each new step of additional facts a child learns is based mainly on the four fundamental steps and only on the four basic concepts contained in them. These four concepts are:

1. The concept of one more, that is, going up by one.
2. The concept of one less, that is, coming down by one.
3. The concept of symmetry (physically), or doubles (mathematically).
4. The concept of uniqueness (physically), or tens and ones (mathematically).

Professor Schindler's discussion was made on the first manuscript.‡ The discussion of this paper about the adoption of the method was made after his comment regarding the integration of this method. As to the development of a total program, it is hoped that the interested teachers and authors will develop such programs or use the method for preparation of their teaching materials. While broad guidelines have been outlined in the second paper for revision of existing books and for writing of new books, some suggestions in advance about the integration of the method in terms of goals are presented here.

Figure 11 shows the steps in the process of a child's learning of addition and subtraction. Step I will prepare a child for Table 8, the first goal to be achieved; and Step II, for Table 9, the second goal to be achieved. In Step I, the child will be trained in counting, in understanding, and, in the four concepts of numbers previously described. In Step II, he will be trained in the fundamental facts and in their application *limited* to the derivatives of the facts only. Step III will train a child in the additional facts as application of the four fundamental steps and the four basic concepts. The accomplishment of goal 1 and goal 2 automatically accomplishes the final goal in basic addition, namely, Table 7, and prepares the child for Step IV, that is, for long addition and subtraction.

Two addition facts of zero and one inverse addition fact are shown in Table 9 without any step number, because their presentation is not totally dependent on the consistent-logic method;§ they may be best presented by an author in a sequence most suitable and effective to his plan of presentation.

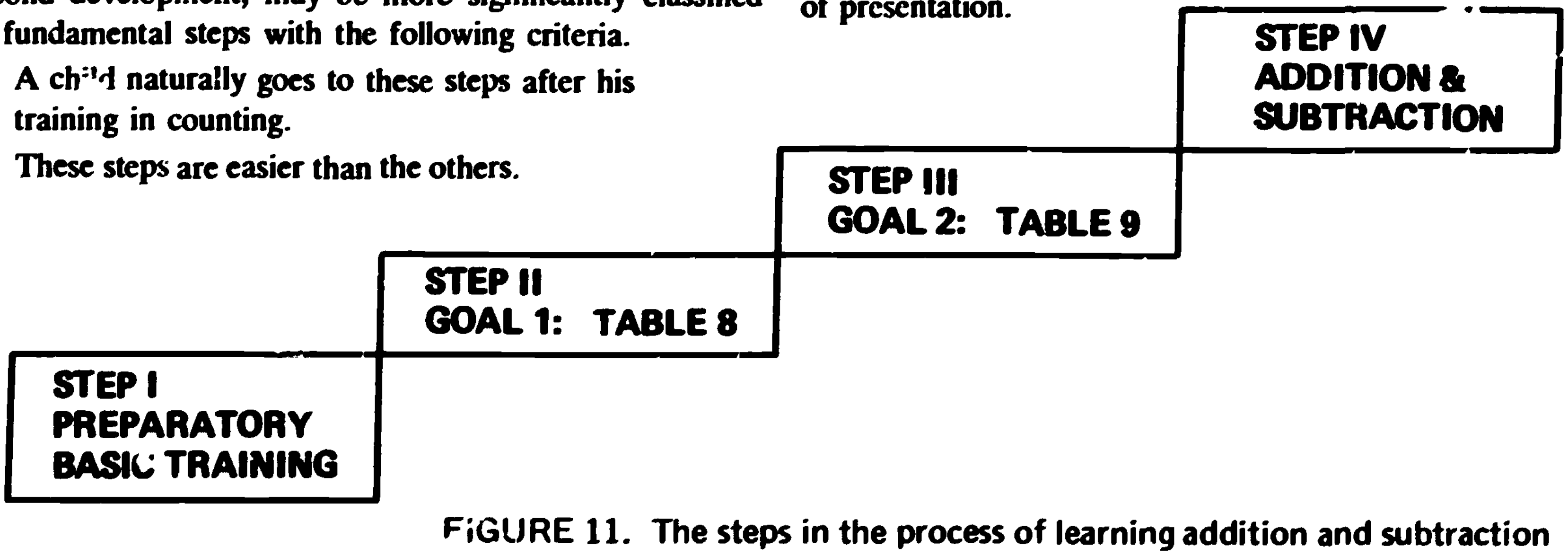

FIGURE 11. The steps in the process of learning addition and subtraction

† a member of the National Council of Teachers of Mathematics

‡ All the discussions were made on the first manuscript.

§ The names, the constant-sum method and the consistent-logic method, were not used in the first manuscript.

STEP 1	1 +1 2	2 +1 3	3 +1 4	4 +1 5	5 +1 6	6 +1 7	7 +1 8	8 +1 9	9 +1 10
STEP 2	2 −1 1	3 −1 2	4 −1 3	5 −1 4	6 −1 5	7 −1 6	8 −1 7	9 −1 8	
STEP 3	2 +2 4	3 +3 6	4 +4 8	5 +5 10	6 +6 12	7 +7 14	8 +8 16	9 +9 18	
STEP 4	10 +1 11	10 +2 12	10 +3 13	10 +4 14	10 +5 15	10 +6 16	10 +7 17	10 +8 18	10 +9 19

TABLE 8. THE FUNDAMENTAL STEPS OF ADDITION

STEP 5	3 +2 5	4 +3 7	5 +4 9	6 +5 11	7 +6 13	8 +7 15	9 +8 17
STEP 6	4 +2 6	5 +3 8	6 +4 10	7 +5 12	8 +6 14	9 +7 16	
STEP 7	9 +2 11	9 +3 12	9 +4 13	9 +5 14	9 +6 15		
STEP 8	5 +2 7	6 +2 8	7 +2 9	8 +2 10			
STEP 9	3 −2 1	4 −2 2	5 −2 3				
STEP 10	8 +3 11	8 +4 12	8 +5 13				2 +8 10
STEP 11	6 +3 9	7 +3 10	7 +4 11			0 +0 0	5 +0 5

TABLE 9. THE ADDITIONAL STEPS OF ADDITION

BY Dr. McNUTT:

Mr. Aziz has applied mathematical logic to the teaching of mathematics in a way so simple, it appeared to me, that it is surprising that it was not done before. He has observed that of the one hundred basic addition facts of decimal arithmetic, a few are particularly easy to learn, and the others can be derived using logical devices understandable to the student.

Eventually the student must commit all the addition facts to memory so that recall is immediate. To make this task easy for a child, new math introduced grouping of basic addition facts and use logic in their derivation. Mr. Aziz, too, groups basic addition facts and uses logic, but in a manner in which logic becomes more systematic and consistent. As a result, I believe, he has made the task of a child simpler and easier. To this end, he also encourages the use of flash cards for drill. In his method, the addition facts are introduced in an order in which sums vary so that subsets of the flash cards can be used before all the facts have been taught.

This work I hope would be considered a start in an important field. What has been done here, I believe, could be extended immediately to addition facts in other number bases and multiplication. Similar techniques involving detailed analysis of materials to be taught may well revolutionize the teaching of elementary mathematics. I would urge teachers to understand not only how a child might find Mr. Aziz's method easier and quicker, but also why he should find it so.

THE AUTHOR'S CLOSURE

I may point out some distinction between committing a fact to memory and deriving it with high speed. To recall the fact $5 + 5 = 10$ is memorization. To recall the result $25 + 15 = 40$ is derivation, and to recall the fact $9 + 8 = 17$ in the new math is also derivation. If the derivation of the fact is so rapid that it seems as if the fact were memorized, there is little distinction in the end between recall by memorization and by rapid derivation. What Dr. McNutt refers to as recall by committing to memory may truly be referred to as recall by rapid derivation for some facts in the new math concepts.

BY PROFESSOR BERG:

I have read Mr. Aziz's paper on teaching addition to children. He suggests an order of presentation of addition facts which should give two benefits. Firstly, it should be possible for the child to learn addition more easily and quickly. Secondly, the child's sense of logic and organization, or thought patterns, should be greatly improved or rapidly developed by this method.

It is easier to learn facts if they are presented in an organized manner. The organization is most helpful if it is directly related to the learning task at hand. Mr. Aziz's paper organizes addition facts in such a way. Rather than using the ordering of the numbers as a principle of organization, Mr. Aziz bases his organization on logic. The facts are divided into two classes—the basic addition facts, and additional addition facts. The basic addition facts are simple, and few in number. The additional addition facts are then grouped into steps (as in the basic facts) and the child progresses one step at a time. Within each step, all facts can be derived from the fundamental facts, *by the same logic*. The child thus is given a method he can use with confidence for all the facts to be learned within the same step. The author points out that such methods of reasoning out the answers are taught in the present method, but the logic is not used in the consistent manner.

This grouping by logic has another advantage. Drilling, by use of flash cards, is a generally accepted manner of increasing speed and proficiency. If, as is presently done, the facts are grouped on the basis of the sum, then obviously the flash method is inoperable. In Mr. Aziz's method it is the logic which is constant and not the sum, thus enabling the use of flash cards.

I hope Mr. Aziz's work would be widely read, as I believe, it would lead to the significant development of teaching of arithmetic in the lower grades.

BY PROFESSOR ELLING:

I read the manuscript of Mr. Aziz' paper with considerable interest. I can recognize the great amount of time, thought and effort he has devoted to this work. I believe, this warrants sincere congratulation.

I find the author's approach to the development of his method creative and ingenious. He studied the past and the present methods with a view to discover a pattern or a clue that could lead to an improvement in the present method. His interpretation of grouping of addition facts in terms of direction, I believe, gave him the clue. He then discovered a direction for grouping in which the use of logic was systematic and consistent.

I have never been directly involved with the teaching of a child, and hence have little knowledge of what the elementary thinking processes are that a child can bring to bear on a subject. I learned addition by the old method, but as best I can remember we did not use finger counting and were still able to master long addition after the first few grades. I have some question about whether imposing a need for exercising a deductive process in the mastering of basic addition is truly an advantage. I have no reason for believing this other than my intuition regarding a child's learning capacity. Suitable tests may well prove the author's point. I think that many people, such as myself, who are not intimately familiar with a child's learning process, may well be skeptical until tests on the basis of class-room instruction are conducted and that results have been shown to be conclusive. My suggestion, therefore, would be to encourage the author to find some means by which these tests can be conducted, even if the initial tests are of limited scope.

Overall, I find the author's ideas very interesting and something that is worth pursuing, for if it were a contribution, it would benefit our children for all generations.

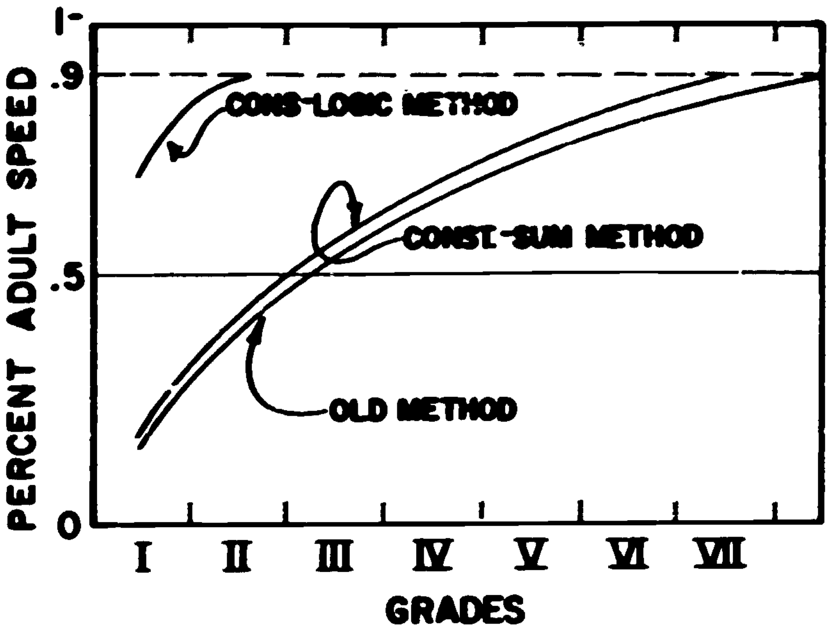

FIGURE 12. Expected progress of learning of addition facts by different methods

THE AUTHOR'S CLOSURE

Comments of Professor Elling were made on the first manuscript in which, I believe, it was not as clear as in the final paper that logic was being employed in the method currently in use (the constant-sum method). The consistent-logic method does not introduce logic, it uses logic systematically and consistently that was introduced by the constant-sum method. We could assume that if logic helped a child in the constant-sum method, using it systematically and consistently should increase its effectiveness. Therefore, Professor Elling's question about the need for logic could rightfully belong to the constant-sum method.

The constant-sum method has been used for many years. It has been found to be an efficient and effective method, and an improvement to the old method to the extent that it has practically replaced the old method. Some reasons were discussed for the adoption of the constant-sum method from an analytical point of view. Other reasons may be discussed from a practical point of view. Let us consider that a child is mastering the fact $9 + 4 = 13$. After the constant-sum method was introduced, he could still master the fact by the old method; but in case he forgot the fact, he could fall back on the derivation if he knew it. Moreover, by the constant-sum method, not only a child knows that $9 + 4 = 13$, but also he knows why. Another point may be brought out. In adopting the constant-sum method, we did not dump the old method, we use the direction of the old method for drill by flash cards. These are the practical reasons for which the constant-sum method was preferred to the old method.

By the old method in this paper was implied a method which did not use logic and derive addition facts. Finger counting was an example. That the finger counting was an example was not clear in the first manuscript. To respond to Professor Elling's comment on the mastery of addition facts, we take the old method he has referred to, and assume that the child's progress in mastering the addition facts by this method is indicated by the curve labelled "old method" in Figure 12. The ordinate of Figure 12 represents the percent average adult speed per fact with which a child can answer the addition problems [cf., "Learning About Numbers" previously cited] or recall the addition facts [cf., Dr. McNutt]. Expected progress of a child by the constant-sum method and the consistent-logic method are also shown in Figure 12. The practical reasons for which the constant-sum method was preferred to the old method has been listed previously. In terms of recall-speed, the constant-sum method may not have an appreciable improvement, or perhaps any, but the consistent-logic method should result in a marked improvement, as shown by the uppermost curve of Figure 12.

Coming to the need for testing the method on the basis of class-room instruction I agree with Professor Elling. Teachers, principals, supervisors and superintendents for elementary schools also have strongly suggested the need for testing the method on the basis of class-room instruction under controlled conditions. I believe the method should be tested for two reasons.

1. To determine a quantitative measure of improvement of the consistent-logic method, or in other words, to determine the progressive values of AB of Figure 12.
2. To demonstrate to practical educators, in a short time and without much analytical effort, that the consistent-logic method is an improvement to the constant-sum method, as concluded.

ACKNOWLEDGMENT

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ABOUT THE AUTHOR

The author holds a Master of Science degree in Mechanical Engineering from Michigan State University. He did further graduate work at MIT, Columbia and Johns Hopkins, and is now working for the doctoral degree at the University of Maryland with an Edison Memorial Scholarship from the Naval Research Laboratory under the work-study plan. He previously worked for the General Electric Company, was an Associate Professor of Mathematics at the Hudson Valley Community College, and an Instructor of Mathematics at the Chicago campus of the Northwestern University. He is a member of the National Council of the Teachers of Mathematics. (The present work was undertaken and conducted by him outside his duties at the Naval Research Laboratory.)

† a member of the National Council of Teachers of Mathematics
‡ a former elementary school teacher

STEPS									
1	1+1=2								
2	2+1=3	1+2=3							
3	3+1=4	2+2=4	1+3=4						
4	4+1=5	3+2=5	2+3=5	1+4=5					
5	5+1=6	4+2=6	3+3=6	2+4=6	1+5=6				
6	6+1=7	5+2=7	4+3=7	3+4=7	2+5=7	1+6=7			
7	7+1=8	6+2=8	5+3=8	4+4=8	3+5=8	2+6=8	1+7=8		
8	8+1=9	7+2=9	6+3=9	5+4=9	4+5=9	3+6=9	2+7=9	1+8=9	
9	9+1=10	8+2=10	7+3=10	6+4=10	5+5=10	4+6=10	3+7=10	2+8=10	1+9=10
10	9+2=11	8+3=11	7+4=11	6+5=11	5+6=11	4+7=11	3+8=11	2+9=11	
11	9+3=12	8+4=12	7+5=12	6+6=12	5+7=12	4+8=12	3+9=12		
12	9+4=13	8+5=13	7+6=13	6+7=13	5+8=13	4+9=13			
13	9+5=14	8+6=14	7+7=14	6+8=14	5+9=14				
14	9+6=15	8+7=15	7+8=15	6+9=15					
15	9+7=16	8+8=16	7+9=16						
16	9+8=17	8+9=17							
17	9+9=18								

TABLE 5. THE ADDITION TABLE
by THE CONSTANT-SUM METHOD

1+1=2	2+1=3	3+1=4	4+1=5	5+1=6	6+1=7	7+1=8	8+1=9	9+1=10
1+2=3	2+2=4	3+2=5	4+2=6	5+2=7	6+2=8	7+2=9	8+2=10	9+2=11
1+3=4	2+3=5	3+3=6	4+3=7	5+3=8	6+3=9	7+3=10	8+3=11	9+3=12
1+4=5	2+4=6	3+4=7	4+4=8	5+4=9	6+4=10	7+4=11	8+4=12	9+4=13
1+5=6	2+5=7	3+5=8	4+5=9	5+5=10	6+5=11	7+5=12	8+5=13	9+5=14
1+6=7	2+6=8	3+6=9	4+6=10	5+6=11	6+6=12	7+6=13	8+6=14	9+6=15
1+7=8	2+7=9	3+7=10	4+7=11	5+7=12	6+7=13	7+7=14	8+7=15	9+7=16
1+8=9	2+8=10	3+8=11	4+8=12	5+8=13	6+8=14	7+8=15	8+8=16	9+8=17
1+9=10	2+9=11	3+9=12	4+9=13	5+9=14	6+9=15	7+9=16	8+9=17	9+9=18

TABLE 3. THE ADDITION TABLE
ACCORDING TO THE OLD METHOD

STEPS									
1	1+1=2	2+1=3	3+1=4	4+1=5	5+1=6	6+1=7	7+1=8	8+1=9	9+1=10
2	2+2=4	3+3=6	4+4=8	5+5=10	6+6=12	7+7=14	8+8=16	9+9=18	
3	3+2=5	4+3=7	5+4=9	6+5=11	7+6=13	8+7=15	9+8=17		
4	4+2=6	5+3=8	6+4=10	7+5=12	8+6=14	9+7=16			
5	9+2=11	9+3=12	9+4=13	9+5=14	9+6=15				
6	5+2=7	6+2=8	7+2=9	8+2=10					
7	8+3=11	8+4=12	8+5=13						
8	6+3=9	7+3=10	7+4=11						

TABLE 7. THE ADDITION TABLE
by THE CONSISTENT-LOGIC METHOD

FIGURE 13. Summary of addition tables

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